Stochastic Processes: Classification

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STOCHASTIC PROCESSES - CLASSIFICATION

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1. DEFINITIONS

Let $\{\Omega, \mathcal{F}, P\}$ be a complete probability space where Ω is the *sample space*, $\mathcal{F}$ is the σ-field associated with the sample space containing all the null sets of Ω, and P is the probability measure defined on the field $\mathcal{F}$. Let $\{\mathbb{R}, \mathcal{R})$ be a measurable range space called the *state space*, where $\mathbb{R} \equiv (-\infty, \infty)$ is the real line and $\mathcal{R}$ is the σ-field associated with the real line $\mathbb{R}$. A *random variable* X is a function that assigns a rule of correspondence between each $\omega \in \Omega$ and each $x \in \mathbb{R}$. This correspondence will induce a probability measure P_X defined on the field $\mathcal{R}$. Thus, X maps the probability space $\{\Omega, \mathcal{F}, P\}$ to the probability range space $\{\mathbb{R}, \mathcal{R}, P_X\}$

$$X : \{\Omega, \mathcal{F}, P\} \longrightarrow \{\mathbb{R}, \mathcal{R}, P_X). \tag{1}$$

The distribution function $F_X(x)$ of X is given by

$$P\{\omega : X(\omega) \leq x\} = P\{X \leq x\} = F_X(x),\ x \in \mathbb{R} \tag{2}$$

and the density function $f_X(x)$, which may include impulse functions of x, is the derivative of $F_X(x)$.

The definition of a *stochastic* (or *random*, or *chance*) process (Gikhman and Skhorokod, 1996, pp. 1, 144) (*see* **Stochastic processes**) requires a parameter set Θ and an increasing sequence of sub σ-fields $\{\mathcal{F}_\theta \subset \mathcal{F}, \theta \in \Theta\}$ called the *filtration σ-field* such that $\mathcal{F}_\zeta \subset \mathcal{F}_\theta$ for each $\{\theta, \zeta \in \Theta, \zeta < \theta\}$. The filtration σ-field is a consequence of the distinction between the uncertainty of the future and the knowledge of the past. The family $\{X(\theta), \mathcal{F}_\theta$ of random variables defined on the probability space $\{\Omega, \mathcal{F}, P\}$ will be called a *random function* if the parameter set Θ is arbitrary and a *stochastic process* if the parameter set Θ is the time set $\mathbb{T} \equiv (-\infty, \infty)$, and θ is interpreted as time t. Thus,

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$X(t) \in \mathcal{F}_t$ is a stochastic process that maps the probability space $\{\Omega, \mathcal{F}, P\}$ to the range space $\{\mathbb{R}, \mathcal{R}, P_X\}$ for every point $\omega \in \Omega$ and $t \in \mathbb{T}$. $X(t)$ is said to be *adapted* to the filtration field $\{\mathcal{F}_t, t \in \mathbb{T}\}$ if $X(t)$ is $\mathcal{F}_t$-measurable in the sense the inverse image set $\{X(t)^{-1}[\mathbb{B}]\} \in \mathcal{F}_t$ for every subset $\mathbb{B}$ of the real line $\mathbb{R} \in \mathcal{R}$.

The important point to emphasize is that a stochastic process is not a single time function but an ensemble of time functions. If the time parameter t belongs to a set of integers $\mathbb{Z} \equiv \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ then $X(n)$ or X_n denotes a *discrete-time* stochastic process.

A non-negative real line will be represented by $\mathbb{R}^+ \equiv [0, \infty)$ and non-negative time set by $\mathbb{T}^+ \equiv [0, \infty)$. A set of non-negative integers will be denoted by $\mathbb{N} \equiv \{0, 1, \ldots\}$ and a set of positive integers by $\mathbb{N}^+ \equiv \{1, 2, \ldots, N\}$.

Since $X(t)$ is a random variable for every $t \in \mathbb{T}$, the distribution function $F_X(x : t)$ will be given by

$$P\{X(\omega, t) \le x\} = P\{X(t) \le x\} \equiv F_X(x : t),\ x \in \mathbb{R},\ t \in \mathbb{T} \tag{3}$$

and the density function $f_X(x : t)$, which again may include impulse functions of x, is the partial derivative of $F_X(x; t)$ with respect to x.

Autocorrelation and autocovariance functions for a stochastic process $X(t)$ for $\{t_1, t_2 \in \mathbb{T}\}$ are defined by:

$$R_X(t_1, t_2) = \big[X(t_1)X(t_2)\big] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2 : t_1, t_2)dx_1 dx_2, \tag{4}$$

$$\begin{aligned} C_X(t_1, t_2) &= E\big\{[X(t_1) - \mu_x(t_1)][X(t_2) - \mu_x(t_2)]\big\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [x_1 - \mu_x(t_1)][x_2 - \mu_x(t_2)] f(x_1, x_2 : t_1, t_2)dx_1 dx_2, \end{aligned} \tag{5}$$

where $\mu_x(t_1)$ and $\mu_x(t_2)$ are the mean values of $X(t)$ at times t_1 and t_2 respectively.

Stochastic processes can be classified in different categories but many of them straddle categories.

2. STATIONARY AND ERGODIC PROCESS

A stochastic process $X(t)$ is n^{th} order *stationary* if the n^{th} order distribution function satisfies

$$F_X(x_1, \ldots, x_n : t_1, \ldots, t_n) = F_X(x_1, \ldots, x_n : t_1 + \tau, \ldots, t_n + \tau) \text{ for any } \tau \in \mathbb{T}. \tag{6}$$

It is *strictly stationary* if Eq. (6) is true for all $n \in \mathbb{Z}$. However, the most useful concepts of stationarity are the first order stationarity defined by

$$F_X(x : t) = F_X(x : t + \tau) = F_X(x), \tag{7}$$

and the second order stationarity called *wide sense stationary* defined by

$$F_X(x_1, x_2 : t_1, t_2) = F_X(x_1, x_n : t_1 + \tau, t_2 + \tau) = F_X(x_1, x_2 : \tau). \tag{8}$$

Wide sense stationarity can be determined from the following two criteria:

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1. The expected value $E[X(t)] = \mu_X =$ a constant.

2. The autocorrelation function $R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau)$ is a function of the time difference τ.

A stationary process $X(t)$ is *mean ergodic* if the ensemble average is equal to the time average of the sample function $X(t)$.

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)dt = \int_{-\infty}^{\infty} x f_X(t)dt = \mu_X, \tag{9}$$

or, equivalently the covariance $C_X(\tau)$ satisfies the condition $\int_{-\infty}^{\infty} |C_X(\tau)|d\tau < \infty$.

A stationary process is *correlation ergodic* if

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2 : \tau)dx_1 dx_2 = R_X(\tau), \tag{10}$$

which is equivalent to the condition $\int_{-\infty}^{\infty} \big|E\{[X(t)X(t+\tau)]^2\} - E\{[X(t)]^2\}\big| d\tau < \infty$.

3. STATE AND TIME DISCRETIZED PROCESS

The stochastic process $X(t)$ can be classified into four broad categories depending upon whether the state space is discretized with $\mathbb{R} \equiv \mathbb{Z}$ or the time is discretized with $\mathbb{T} \equiv \mathbb{Z}$ or both. As mentioned earlier, discrete-time stochastic processes will be denoted by X_n or $X(n)$ where $n \in \mathbb{Z}$.

1. *Discrete State Discrete Time Process* (DSDT)

At any given time $i > 0$ a particle takes a positive step from $X_0 = 0$ with probability p and a negative step with probability q with $p + q = 1$. The random variable Z_i representing each step is independent and identically distributed. The position X_n of the particle at time n is a stochastic process $X_n = Z_1 + Z_2 + \ldots + Z_n$. It represents a DSDT process with discrete time set $\mathbb{N}^+ = \{1, \ldots, n, \ldots\}$ and discrete state space $\mathbb{R} = \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ representing the position of the particle. This process known as a *simple random walk* (Cox and Miller, 1977, p. 25) is nonstationary. If $p = q$ then the process is called a *symmetric* simple random walk.

2. *Discrete State Continuous Time Process* (DSCT)

A customer arrives at the service counter of a supermarket at a random time $t \geq 0$ at an average rate of λ per unit time interval. If $N(t)$ is the stochastic process representing the number of customers arriving in the time interval $[0, t]$ then $N(t)$ is a DSCT process with time set $\mathbb{T}^+ = \{0 \leq t < \infty\}$ and discrete state space $\mathbb{N} = \{0, 1, \ldots\}$ representing the number of customers. This process known as *Poisson process* is nonstationary.

3. *Continuous State Discrete Time Process* (CSDT)

In the DSDT process of (1), each step of the particle at any time $i > 0$ is a continuous random variable Z instead of a discrete one, governed by a distribution

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function $F_Z(z)$ with mean μ_Z. If X_n is the position of the particle at time $i = n$ then X_n represents a CSDT process with discrete time set $\mathbb{N}^+ = \{1, \ldots, n, \ldots\}$, and continuous state space $\mathbb{R}^+ = \{0 \leq x < \infty\}$ representing the position of the particle. This process is nonstationary.

4. *Continuous State Continuous Time Process* (CSCT)

In the DSDT process of (1) the particle undergoes a positive or negative step of Δx in a time interval Δt. If certain limiting conditions on Δx and Δt are satisfied then as Δx and Δt tend to 0, a CSCT process results, which is called *Wiener process* (Cox and Miller, 1977, p. 205) or *Brownian motion*. Extrusion of plastic shopping bags where the thicknesses of the bags vary constantly with respect to time with the statistics being constant over long periods of time is an example of a CSCT process. These processes are nonstationary.

4. GAUSSIAN PROCESS

A stochastic process $X(t)$ defined on a complete probability space is a *Gaussian process* if for any collection of times $\{t_0, t_1, \ldots, t_n\} \in \mathbb{T}$, the random variables $X_0 = X(t_0), X_1 = X(t_1), \ldots, X_n = X(t_n)$ are jointly Gaussian distributed for all $n \in \mathbb{Z}$, with joint probability function

$$f_{X_0 X_1 X_2 \ldots X_n}(x) = \frac{1}{(2\pi)^{n/2}|\mathbf{C}_X|} \exp\left(-\frac{(\mathbf{x}-\boldsymbol{\mu}_X)^T \mathbf{C}_X^{-1}(\mathbf{x}-\boldsymbol{\mu}_X)}{2}\right) \tag{11}$$

where $\boldsymbol{\mu}_X$ is the mean vector and $\mathbf{C}_X$ is the covariance matrix of the random variables $\{X_0, X_1, \ldots, X_n\}$. The Wiener process is also an example of a Gaussian process.

5. MARKOV PROCESS

Let the σ-field $\mathcal{F}_t$ generated by $\{X(s), s \leq t, t \in \mathbb{T}\}$ represent the past history up to the present and the σ-field $\mathcal{F}_t^c$ generated by $\{X(s), s > t, t \in \mathbb{T}\}$ represent the future evolution. Let a random variable Y be $\mathcal{F}_t$-measurable and another random variable Z be $\mathcal{F}_t^c$-measurable. Then the process $\{X(t), t \in \mathbb{T}\}$ is called a *Markov process* if the following hold:

1. Given the present information $X(t)$, the past Y and the future Z are conditionally independent.

$$E[YZ|X(t)] = E[Y|X(t)]E[Z|X(t)]. \tag{12}$$

2. The future Z, conditioned on the past history up to the present $\mathcal{F}_t$, is equal to the future given the present.

$$E[Z|\mathcal{F}_t] = E[Z|X(t)]. \tag{13}$$

3. The future Z, conditioned on the past value $X(s)$ is the future conditioned on the present value $X(t)$ and again conditioned on the past value $X(s)$.

$$E[Z|X(s)] = E\{E[Z|X(t)]|X(s)\} \text{ for } s < t. \tag{14}$$

Invited chapter published in the International Encyclopedia of Statistical Science, Springer, December 2010, pp. 1541-1547

This is known as the *Chapman-Kolmogorov equation* (Ross, 2000. p. 166).

In terms of probability, with $\tau > 0$ and states x_h, x_i, x_j, Eq. (13) is equivalent to:

$$P\{X(t+\tau) = x_j | X(t) = x_i, X(u) = x_h, 0 \le u < t\} = P\{X(t+\tau) = x_j | X(t) = x_i\}. \quad (15)$$

Or, for $t_0 < t_1 < \ldots < t_{n-1} < t_n$, and $\{x_k, k = 0, \ldots, n, \ldots\}$ belonging to some discrete-state space

$$\begin{aligned} P\{X(t_{n+1}) = x_{n+1} | X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots, X(t_0) = x_0\} \\ = P\{X(t_{n+1}) = x_{n+1} | X(t_n) = x_n\}. \end{aligned} \quad (16)$$

A Markov process has an important property that the density $f_{\tau_i}(t)$ of the random time τ_i spent in any given state x_i is an exponential and hence it is called *memoryless.*

6. MARKOV CHAINS

Discrete state Markov processes are called *chains*, and if time is continuous they are called *continuous Markov chains*, and if time is discrete they are called *discrete Markov Chains.* The Poisson process is an example of a continuous Markov chain.

A stochastic process $\{X(t), t \in \mathbb{T}^+\}$ is a continuous-time Markov chain if for each of the discrete states h, i, j and any time $\tau > 0$

$$P\{X(t+\tau) = j | X(t) = i, X(u) = h, 0 \le u < t\} = P\{X(t+\tau) = j | X(t) = i\}. \quad (17a)$$

The quantity $P\{X(t+\tau) = j | X(t) = i\}$ is the time dependent transition probability defined by $p_{ij}(t, \tau)$, which is generally a function of times t and τ. If the transition from the state i to the state j is dependent only on the time difference $\tau = (t + \tau) - t$ then the transition probability is stationary and the Markov chain is called *homogeneous.* In this case transition probability becomes $p_{ij}(\tau)$.

The probability density function $f_{\tau_i}(t)$ of the random time τ_i spent in any given state i for a continuous Markov chain is exponential and hence it is called *memoryless.*

A stochastic process $\{X(n), n = 0, 1, \ldots\}$ is a discrete-time Markov chain if for each of the discrete states i, j and $\{i_k, k = 0, 1, \ldots, n-1\}$ and any time $m > 0$,

$$\begin{aligned} P\{X(n+m) = j | X(n) = i, X(n-1) = i_{n-1}, \ldots, X(0) = i_0\} \\ = P\{X(n+m) = j | X(n) = i\}. \end{aligned} \quad (17b)$$

The quantity $P\{X(n+m) = j | X(n) = i\}$ is called the m-step transition probability defined by $p_{ij}^{(m)}(n)$, which is generally a function of time n. If the transition from the state i to the state j is dependent only on the time difference $m = (n+m) - n$ then the transition probability is stationary and the Markov chain is *homogeneous.* In this case the m-step transition probability becomes $p_{ij}^{(m)}$.

Invited chapter published in the International Encyclopedia of Statistical Science, Springer, December 2010, pp. 1541-1547

The one-step probability from state i to state j of a homogeneous discrete Markov chain is given by:

$$P\{X(n+1) = j | X(n) = i\} = p_{ij}. \quad (18)$$

The probability mass function f_{τ_i} of the random time τ_i spent in any given state i for a discrete Markov chain is geometric and hence it is called *memoryless*.

7. SEMI-MARKOV PROCESS

In a Markov process the distributions of state transition times are exponential for a continuous process, and geometric for a discrete process and hence they are considered memoryless. While the definition of a *semi-Markov process* $X(t)$ defined on a complete probability space is the same as that of a Markov process (Eqs. 15, 16), the distributions of transition times $\tau_i \in \mathbb{T}$ between states need not be memoryless but can be arbitrary. For a continuous-time semi-Markov process the state transitions can occur at any instant of time $t \in \mathbb{T}$ with an arbitrary density $f_{\tau_i}(t)$ for the time τ_i spent in state x_i and for a discrete-time semi-Markov process the state transitions can occur at time instants $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ with an arbitrary probability mass f_{τ_i} for the time τ_i spent in state i. If the amount of time spent in each state is 1 then this semi-Markov process is a Markov chain. Markov processes are a subclass of semi-Markov processes.

8. INDEPENDENT INCREMENT PROCESS

A stochastic process $\{X(t), t \in \mathbb{T}\}$ is defined on a complete probability space with a sequence of time variables $\{t_0 < t_1 < \ldots < t_n\} \in \mathbb{T}$. If the increments $X(t_0), [X(t_1) - X(t_0)], \ldots, [X(t_n) - X(t_{n-1})]$ of the process $\{X(t), t \in \mathbb{T}\}$ are a sequence of independent random variables then the process is called an *independent increment* (Krishnan, 2006, p. 507) process. If the distribution of the increments $X_t - X_s$, $t > s$ depends only on the time difference $t - s = \tau$, then the process is a *stationary independent increment* process.

If the time set is discrete given by $\mathbb{N}^+ = \{1, 2, \ldots\}$ then the independent increment process is a sequence of independent random variables given by $Z_0 = X_0, \{Z_i = X_i - X_{i-1}, i \in \mathbb{N}^+\}$. Independent increment process is a special case of a Markov process. It is not a stationary process because of the following (Krishnan, 2005, p. 61: 2006, p. 507):

$$E[X(t)] = \mu_0 + \mu_1 t, \text{ where } \mu_0 = E[X(t_0)] \text{ and } \mu_1 = E[X(t_1)] - \mu_0;$$

$$\text{Var}[X(t)] = {\sigma_0}^2 + {\sigma_1}^2 t, \text{ where } {\sigma_0}^2 = E[X(t_0) - \mu_1]^2 \text{ and } {\sigma_1}^2 = E[X(t_1) - \mu_0]^2 - {\sigma_0}^2. \quad (19)$$

Poisson and Wiener processes are examples of stationary independent increment processes.

Uncorrelated and Orthogonal Increment Process

A stochastic process $\{X(t), t \in \mathbb{T}\}$ with $s_1 < t_1, s_2 < t_2$ and $t_1 \le t_2$

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1. Has *uncorrelated increments* (Krishnan, 2006, p. 508) if

$$E[(X_{t_2} - X_{s_2})(X_{t_1} - X_{s_1})] = E[(X_{t_2} - X_{s_2})]E[(X_{t_1} - X_{s_1})]. \tag{20}$$

2. Has *orthogonal increments* (Krishnan, 2006, p. 508) if

$$E[(X_{t_2} - X_{s_2})(X_{t_1} - X_{s_1})] = 0. \tag{21}$$

Clearly, independent increments imply uncorrelated increments but the converse is not true.

9. GENERAL RANDOM WALK PROCESS

The simple random walk discussed earlier can be generalized. Starting from $X_0 = 0$ a particle takes independent identically distributed random steps $Z_1, Z_2, \ldots,$ Z_n, whose values are drawn from an arbitrary distribution, which do not change with the state of the process. This distribution may be continuous with density function $f_Z(z)$ or discrete with probability of transition from state i to state j being p_{ij}. In the latter case p_{ij} will be dependent on the difference $j - i$, or, $p_{ij} = p_{j-i}$. The position $X_n = Z_1 + Z_2 + \ldots + Z_n$, $n \in \mathbb{N}^+$ of the particle is a stochastic process where n is the number of state transitions, which is always forward from state x_i to x_{i+1}. Depending upon whether the instants of these transitions are taken from the set $\mathbb{T}^+$ or $\mathbb{N}^+$ the process X_n is either a continuous-time or a discrete-time *general random walk* (Cox and Miller, 1997, p. 46). In either case the distribution of the time intervals between these transitions is arbitrary and hence it is a special case of a semi-Markov process.

10. BIRTH AND DEATH PROCESS

Let $\{X(t), t \geq 0\}$ be a continuous Markov chain. State transitions can occur only from the state $x_i = i$ to $x_{i+1} = i + 1$, or $x_{i-1} = i - 1$, or stays at $x_i = i$. $X(t)$ is called a *birth and death process* (Kleinrock, 1975, p.53) if in a small interval Δt

$$P\{X(t + \Delta t) - X(t) = j | X(t) = i\} = \begin{cases} \lambda_i \Delta t + o(\Delta t), & \text{if } j = 1, \\ \mu_i \Delta t + o(\Delta t), & \text{if } j = -1, \\ o(\Delta t), & \text{if } |j| > 1. \end{cases} \tag{22}$$

$$\text{and } P\{X(t + \Delta t) - X(t) = 0 | X(t) = i\} = 1 - (\lambda_i + \mu_i)\Delta t + o(\Delta t), \tag{23}$$

where $o(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$. λ_i is the rate at which births occur and μ_i is the rate at which deaths occur when the population size is i. The probability of the population size being i at any time $t > 0$ is given by $P\{X(t) = i\} = P_i(t)$. This is a Markov process with independent increments. If $\lambda_i = i\ \lambda$ and $\mu_i = i\ \mu$ then this process is called a *linear* birth and death process.

The *pure birth process* is a sub-class of birth and death process with $\mu_i \equiv 0$ for all i. State transitions can occur only from the state $x_i = i$ to $x_{i+1} = i + 1$ with rate λ_i or stays in the same state $x_i = i$.

The *Poisson process* is a sub-class of pure birth processes with $\lambda_i \equiv \lambda$ a constant for all i. Here the probability of i events in time t is given by $P_i(t, \lambda) = [(\lambda t)^i / i!]e^{-\lambda t}, t > 0$. This process has stationary independent increments.

Invited chapter published in the International Encyclopedia of Statistical Science, Springer, December 2010, pp. 1541-1547

11. RENEWAL PROCESS

In the general random walk process X_n discussed in the previous section the interest was in the probability of the state of the particle after n transitions. In renewal processes the concern is only in the number of transitions that occur in a time interval $[0, t]$ and not on the state. Starting from $t = 0$ the transitions occur at sequence of times $0 < t_1 < t_2 < \ldots < t_n, n > 0$ with *inter-arrival times* defined by random variables $Y_1 = t_1, Y_2 = (t_2 - t_1), \ldots, Y_n = (t_n - t_{n-1})$. The random variables $Y_i, i \in \mathbb{N}^+$ are independent and identically distributed with an arbitrary density function $f(y)$ with $E[Y_i] = \mu$ for all i.

The stochastic process defined by $X_n = Y_1 + Y_2 + \ldots + Y_n$ is called a *renewal* process (Cox and Miller, 1977, p. 340), where a renewal occurs at the epochs at $t_1 < t_2 < \ldots < t_n$. In this process X_n represents the *time* of the n^{th} renewal whereas in the random walk X_n represents the *state* of the process at time n. This process is a subclass of semi-Markov processes and also a subclass of random walk processes. If the density function $f(y)$ is either exponential or geometric then this process is Markov. The relationship among the various discrete-state stochastic processes similar to the one in Kleinrock, 1975, p.25 is shown in Figure 1.

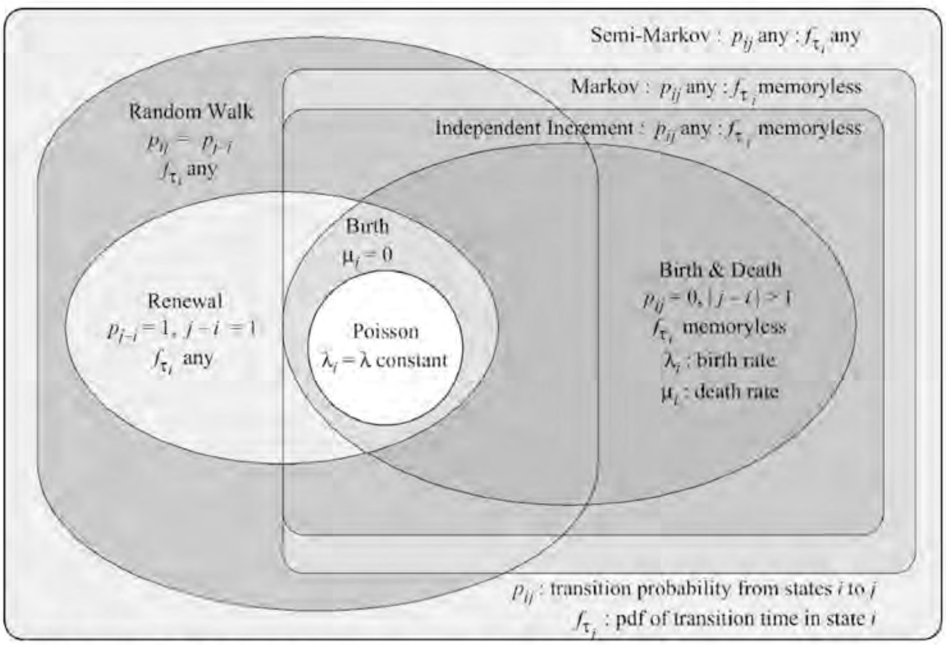

Figure 1 Relationships among some discrete-state stochastic processes

12. MARTINGALE PROCESS

A *martingale process* (Doob, 1990, pp. 91, 294) is a stochastic process where the best estimate of the future value conditioned on the past history including the present is the present

Invited chapter published in the International Encyclopedia of Statistical Science, Springer, December 2010, pp. 1541-1547

value. Since there is no trend to the process it is unpredictable. Many problems in engineering and finance can be cast in the martingale framework. Pricing stock options (Ross, 2000, p. 556) and bonds has been cast in the martingale framework.

Let $\{\Omega, \mathcal{F}, P\}$ be a complete probability space and let $\{\mathcal{F}_n, n \in \mathbb{N}\}$ be an increasing family of sub σ-fields of $\mathcal{F}$. The real valued sequence of random variables $\{X_n, n \in \mathbb{N}\}$ adapted to the family $\{\mathcal{F}_n, n \in \mathbb{N}\}$ is a discrete $\mathcal{F}_n$-martingale if for all n:

1. $E|X_n| < \infty$,
2. $E\{X_n|\mathcal{F}_m\} = X_m$ for $m \leq n$.

If condition (2) is modified as

3. $E\{X_n|\mathcal{F}_m\} \geq X_m$ for $m \leq n$ submartingale,
4. $E\{X_n|\mathcal{F}_m\} \leq X_m$ for $m \leq n$ supermartingale.

Analogously, let $\{\mathcal{F}_t, t \in \mathbb{T}^+\}$ be an increasing family of sub σ-fields of $\mathcal{F}$ of a complete probability space. The real valued stochastic process $\{X(t), t \in \mathbb{T}^+\}$ adapted to the family $\{\mathcal{F}_t, t \in \mathbb{T}^+\}$ is a continuous $\mathcal{F}_t$-martingale if for all $t \in \mathbb{T}^+$:

1. $E|X(t)| < \infty$,
2. $E\{X(t)|\mathcal{F}_s\} = X_s$ for $s \leq t$.

If condition (2) is modified as

3. $E\{X(t)|\mathcal{F}_s\} \geq X_s$ for $s \leq t$ submartingale.
4. $E\{X(t)|\mathcal{F}_s\} \leq X_s$ for $s \leq t$ supermartingale.

Note that any martingale is both a submartingale and a supermartingale.

In the simple random walk process given in DSDT, if $n(p-q)$ is subtracted from X_n, then $Y_n = [X_n - n(p-q)]$ is an example of a discrete martingale with respect to the sequence $\{Z_k, k = 1, \ldots, n-1\}$ even though X_n is not. The Wiener process $W(t)$ is an example of a continuous $\mathcal{F}_t$-martingale. In the Poisson process $N(t)$, if the mean λt is subtracted then $Y(t) = [N(t) - \lambda t]$ is another example of a continuous $\mathcal{F}_t$-martingale even though $N(t)$ is not. However, both X_n and $N(t)$ are Markov processes leading to the conclusion that a Markov process is not necessarily a martingale. It can also be shown that a martingale is not necessarily a Markov process.

The martingale property captures the notion of a fair game. A fair coin is tossed and a player wins a dollar if the toss is heads and loses a dollar if the toss is tails. At the end of the m^{th} toss the player has X_m dollars. The estimated amount of money after the $m + 1^{\text{st}}$ toss is still X_m dollars since the expected value of the $m + 1^{\text{st}}$ toss is zero.

Invited chapter published in the International Encyclopedia of Statistical Science, Springer, December 2010, pp. 1541-1547

13. PERIODIC PROCESS

Let $\{X(t), t \in \mathbb{T}\}$ be a stochastic process defined on a complete probability space taking values in the range space $\{\mathbb{R}, \mathcal{R}\}$. $X(t)$ is *periodic in the wide sense* (Krishnan, 2006, p. 558) with period $T_c(T_c > 0)$ if the mean $\mu_X(t)$ and the autocorrelation function $R_X(t,s)$ satisfy

$$\mu_X(t) = \mu_X(t + kT_c) \text{ for all } t \text{ and integer } k \tag{24}$$

$$R_X(t,s) = R_X(t + kT_c, s) = R_X(t, s + kT_c) \text{ for all } t, s \text{ and integer } k. \tag{25}$$

Note that $R_X(t,s)$ is periodic in both arguments t and s.

However, for a stationary periodic process $X(t)$ with $\tau = t - s$, Eq.(25) simplifies to

$$R_X(\tau) = R_X(\tau + kT_c) \text{ for all } \tau \text{ and integer } k. \tag{26}$$

Since $R_X(\tau)$ is uniformly continuous, a zero mean stationary periodic stochastic process $X(t)$ with fundamental frequency $\omega_c = 2\pi/T_c$ can be represented in the mean square sense by a Fourier series

$$X(t) = \sum_{n=-\infty}^{\infty} X_n \exp(jn\omega_0 t), X_0 = 0 \text{ where } X_n = \frac{1}{T_c}\int_0^{T_c} X(t)\exp(-jn\omega_0 t)dt. \tag{27}$$

Cyclostationary process

Allied to the periodic process is the *cyclostationary process* (Krishnan, 2006, p. 560). A *strict sense* cyclostationary process $X(t)$ on a complete probability space with period $T_c(T_c > 0)$is defined by

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = F_X(x_1, \ldots, x_n; t_1 + kT_c, \ldots, t_n + kT_c) \tag{28}$$

for all n and k.

Since the above definition is too restrictive, a *wide sense* cyclostationary $X(t)$ can be defined by

$$\begin{aligned} \mu_X(t) &= \mu_X(t + kT_c) \\ R_X(t_1, t_2) &= R_X(t_1 + kT_c, t_2 + kT_c). \end{aligned} \tag{29}$$

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